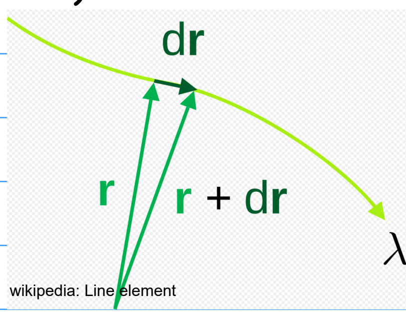
Tutorial Quiz Week 7

1. Calculate the line element, area element and volume element of integration in polar, cylindrical polar, and spherical polar coordinates, respectively. i.e what is?
   - dL in terms of r
   - dA in terms of (r, θ)
   - dV in terms of (r, θ, ϕ)

1^{st}, what is a line/area/volume element?



wikipedia: Line element

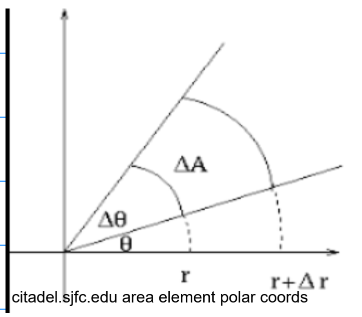
They are a means for integrating a function wrt to length/area/volume respectively.

i.e Integrating over a length of curve:

$$\int dL = \int dr$$

integrating over an area (polar)

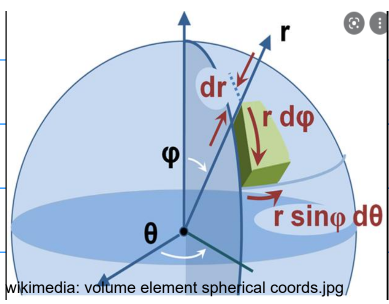
$$\iint dA = \iint r\,dr\,d\theta$$



citadel.sjfc.edu area element polar coords

integrating over a volume (spherical)

$$\iiint dV = \iiint r\sin^2\phi\,dr\,d\theta\,d\phi$$



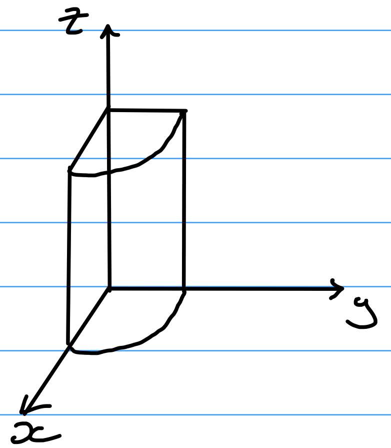
wikimedia: volume element spherical coords.jpg

2. Change the order of integration [illegible] to $dzdydx$ of the following

- $\int_0^5 \int_0^2 \int_0^{\sqrt{4-y^2}} dxdydz$
- $\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} dxdzdy$

a) $\int_0^5 \int_0^2 \int_0^{\sqrt{4-y^2}} dxdydz$. This denotes the region bound by the lines

$x=0$, $x^2 = 4-y^2$, $y=0$, $z=0$, $z=5$

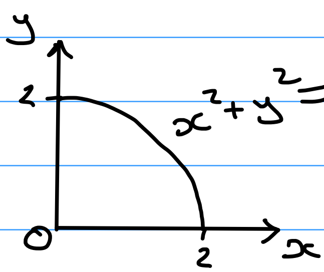


so changing order, z is still constant

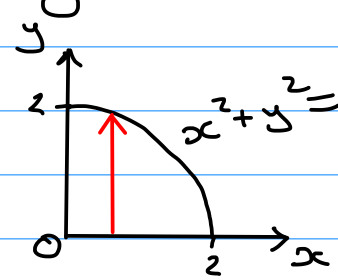
$$\iint_R \left(\int_0^5 dz\right) dA$$

we now change from $dxdy$ to $dydx$

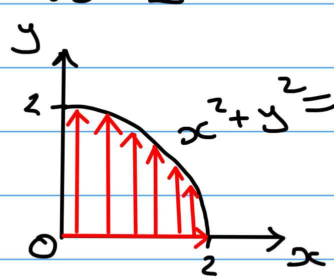
The xy plane has:



So looking at y first it goes from 0 to $y = \sqrt{4-x^2}$



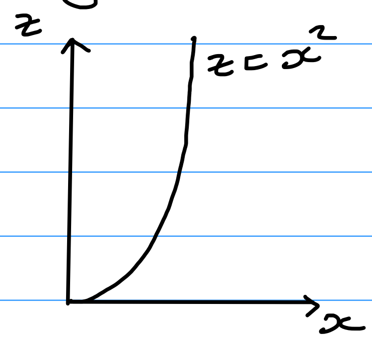
Then expand out along values of x goes from 0 to 2

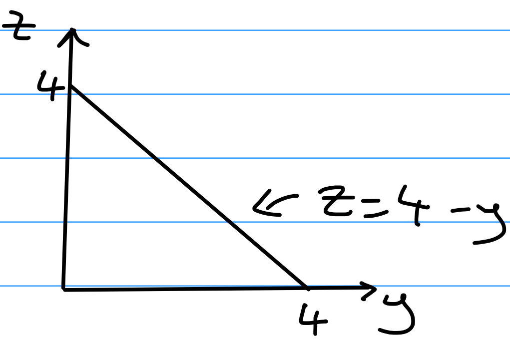


Hence $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^5 dzdydx$

2b) $\displaystyle\int_0^4\int_0^{4-y}\int_0^{\sqrt{z}} dx\,dz\,dy$

Region of integration bound by $x=0$, $z=x^2$, $z=4-y$, $z=0$, $y=4$, $y=0$

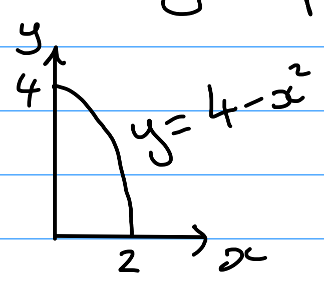




So +ve half of a parabola thats upper limit is the line $4-y$.

Integrating wrt z first the limits are x^2 and $4-y$ $\displaystyle\int_{x^2}^{4-y}$

In the xy plane we have a top down view



This gives $0\le y\le 4-x^2$ and $0\le x\le 2$

Hence $\displaystyle = \int_0^2\int_0^{4-x^2}\int_{x^2}^{4-y} dz\,dy\,dx$

3. Use double integrals in polar coordinates to find the volume of the oblate spheroid $\frac{x^2}{a}+\frac{y^2}{a}+\frac{z^2}{c}=1$ where $0<c<a$

First we need the function in terms of z, then we want the boundary in xy plane.

Rearranging eqtn:

$$\frac{z^2}{c} = 1 - \frac{x^2+y^2}{a}$$

$$z = \pm\sqrt{c - \frac{c}{a}(x^2+y^2)} = \sqrt{\frac{c}{a}}\sqrt{(a-(x^2+y^2))}$$

Take +ve and multiply integral by 2 to get full sphere.

$$V = 2\iint_R \sqrt{\frac{c}{a}}\sqrt{(a-(x^2+y^2))}\,dx\,dy$$

Convert to polar: $x = r\cos\theta$, $y = r\sin\theta$

$$V = 2\sqrt{\frac{c}{a}}\iint_R \sqrt{a-r^2}\,r\,dr\,d\theta$$

In xy plane is a circle $\leftarrow x^2+y^2=a$ so $0 \leq r \leq \sqrt{a}$
$0 \leq \theta \leq 2\pi$

$$V = 2\sqrt{c/a}\int_0^{2\pi}\int_0^{\sqrt{a}} \sqrt{a-r^2}\,r\,dr\,d\theta$$

$$V = 2\sqrt{\frac{c}{a}} \int_0^{2\pi} -\frac{(a-r^2)^{3/2}}{3} \Big|_0^{\sqrt{a}} \, d\theta$$

$$= 2\sqrt{\frac{c}{a}} \int_0^{2\pi} \frac{a^{3/2}}{3} \, d\theta$$

$$= \frac{2\sqrt{c}\,a}{3} \cdot 2\pi = \frac{4\pi}{3} a\sqrt{c}$$

4. Sketch the regions and express them as limits of a triple integral of volume:

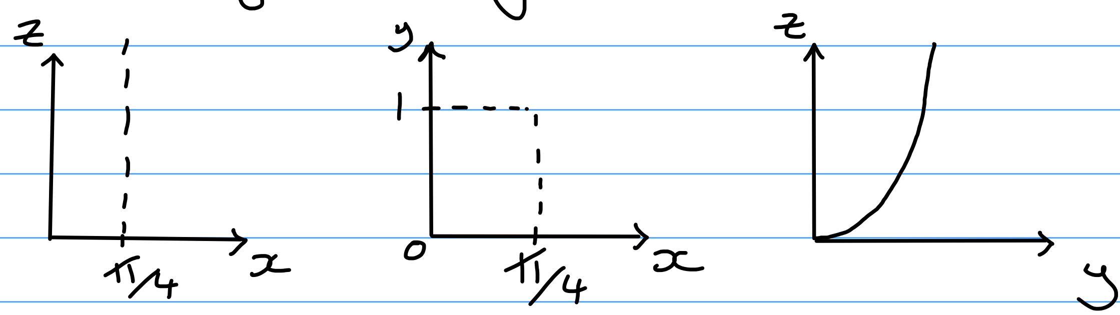
a ∘ $0 \leq z \leq y^2, \quad 0 \leq y \leq 1, \quad 0 \leq x \leq \pi/4$

b ∘ $x \geq 0, \quad y \geq 0, \quad 0 \leq z \leq 1 - y - x$

c ∘ $x^2 + y^2 = 1, \quad z = 0, \quad z = y$

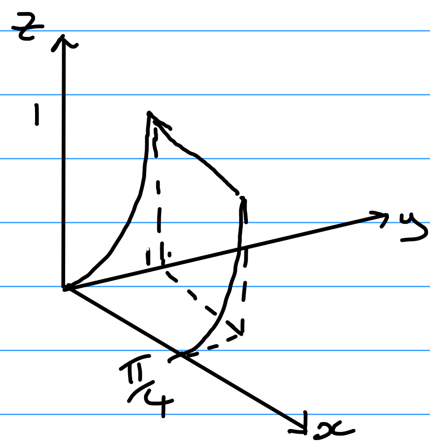
A triple integral volume just means the integrand is 1.

4a) $0 \leq z \leq y^2$, $0 \leq y \leq 1$, $0 \leq x \leq \pi/4$



So 3D:

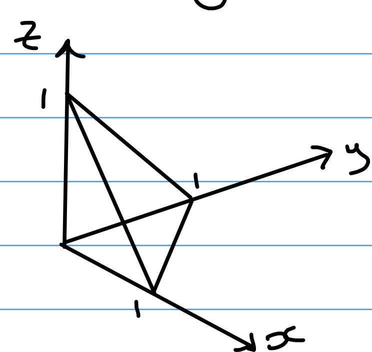
Kind of like a half half pipe.



$$\int_0^{\pi/4} \int_0^1 \int_0^{y^2} dz\,dy\,dx$$

b) $x \geq 0$, $y \geq 0$, $0 \leq z \leq 1 - y - x$

So +ve quadrant, and $z = 1 - y - x$ makes a flat plane that goes through $z = 1 - y$ and $z = 1 - x$

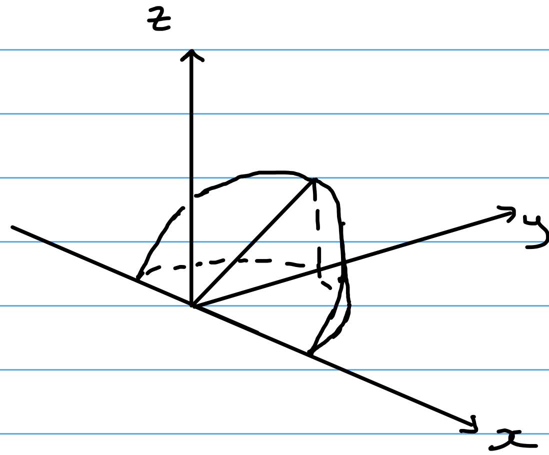


4c $x^2+y^2=1$, $z=0$, $z=y$

The first eqtn is a circle of radius 1, the second two are the volume bounds.

$$0 \le z \le y.$$

So it kind of looks like a wedge.

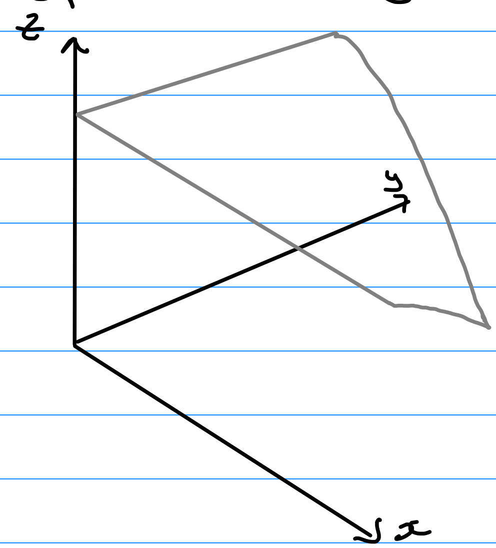


5. Evaluate $\iiint_R 3-4x\,dV$ where R is the region below $z=1-xy$ and above the region in the xy plane defined by $0\leq x\leq 1, 0\leq y\leq 1$.

- Sketch the region
- Determine limits and order of integration
- Evaluate integral

a) $z = 1-xy$ is a saddle thats extended from the hyperbola $xy=1$

(hyperbolic paraboloid



b) Integrate wrt z first : $\int_0^1\int_0^1\int_0^{1-xy} 3-4x\,dz\,dy\,dx$

$$I = \int_0^1\int_0^1 (3-4x)(1-xy)\,dy\,dx$$

$$= \int_0^1 3y - \frac{3}{2}xy^2 - 4xy + 2x^2y^2\Big|_0^1\,dx$$

$$= \int_0^1 3 - \frac{11}{2}x + 2x^2\,dx$$

$$= 3x - \frac{11x^2}{4} + \frac{2x^3}{3}\Big|_0^1 = \frac{11}{12}$$